ORTHOGONAL POLYNOMIALS AND CONTRAST MATRICES

BY

Anna N. Angelos and W. T. Federer

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Abstract

The use of Legendre, Hermite, Chebyshev and Laguerre polynomials in fitting a regression equation and in formulating orthogonal contrast matrices is considered. Two results concerning the use of the orthogonal polynomials in these areas are stated and proved as theorems. Several other types of polynomials are discussed as possibilities for further study.

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1. Introduction

Orthogonal sets of single degree of freedom contrasts between estimated population parameters can play an important role in statistical data analysis. Likewise, fitting a function of one or more variables (X_i) to a response variable Y_i is also an important component of the statistical methods employed in data analysis.

Three frequently used orthogonal contrast matrices are the:

(i) Helmert matrices

(ii) Orthogonal polynomials, and

(iii) Hadamard matrices.

Sivara (1978) considered a general form of an orthogonal contrast matrix of order h as shown in (1.1), and obtained a general form for h = 3, 4 and 5.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_h \\ b_1 & b_2 & b_3 & \cdots & b_h \\ c_1 & c_2 & c_3 & \cdots & c_h \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_h \end{bmatrix} \qquad (1.1)$$

Helmert, orthogonal polynomial, and Hadamard contrast matrices may be obtained from (1.1) by letting the coefficients take on specified values. Other orthogonal contrast matrices used in the analysis of data may also be obtained from (1.1).

A question that arises is: Given the many forms of orthogonal polynomials in mathematical literature, are any of them useful in constructing orthogonal contrast matrices? For pedagogical and data analysis purposes, it would be useful to have additional specific forms of (1.1).

Another question that arises pertains to fitting a response equation $Y_i = f(X_i)$ to a set of data. The most frequently used form of the fitted regression equation in statistical methodology is the form

$$Y_i = f(X_i) = \sum_{j=0}^{h} b_j X_i^j \quad . \tag{1.2}$$

The matrix equation for n observations is

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_1 & X_1^2 & \cdots & X_1^h \\ 1 & X_2 & X_2^2 & \cdots & X_2^h \\ 1 & X_3 & X_3^2 & \cdots & X_3^h \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_n & X_n^2 & \cdots & X_n^h \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix} = \underline{\underline{XB}} \tag{1.3}$$

for $h \leq n-1$.

The orthogonal polynomial coefficients of the X_i are obtained by applying the Gram-Schmidt process to successive columns of the X matrix. This is referred to as the Gram-Schmidt orthogonalization of the X matrix.

A regression equation of the form (1.2) can be generalized as a polynomial

$$Y_i = P(X_i) = \sum_{j=0}^{h} A_j P_j(X_i) \tag{1.4}$$

where $P_j(X_i)$ are functions of the X_i and the A_j are constants. Equations (1.2), (1.3), and (1.4) are termed <u>type 1</u> equations.

A type 2 equation is given as:

$$Y_i = \sum_{j=0}^{h} B_j \frac{R(X_i)}{Q(X_i)} \tag{1.5}$$

where the $R(X_i)$ and $Q(X_i)$ are functions of the X_i and the B_j's are constants. Of course, when $Q(X_i) = 1$, or when $R(X_i)/Q(X_i) = P(X_i)$, (1.5) reduces to (1.4) the type 1 response equation.

A type 3 equation is obtained in the following manner:

$$Y_i = F(X_{1i}, X_{2i}) = \sum_{i=0}^{s-1} \sum_{j=0}^{k-1} b_{ij} X_1^i X_2^j \tag{1.6a}$$

$$Y_i = F(X_{1i}, X_{2i}, X_{3i}) = \sum_{f=0}^{m-1} \sum_{i=0}^{s-1} \sum_{j=0}^{k-1} b_{fij} X_1^f X_2^i X_3^j \tag{1.6b}$$

$$Y_i = F(X_{1i}, X_{2i}, X_{3i}, \cdots, X_{hi}) = \sum_{f=0}^{m-1} \sum_{i=0}^{s-1} \cdots \sum_{j=0}^{k-1} b_{fi\ldots j} X_1^f \cdots X_h^j \quad . \tag{1.6c}$$

Here, the $X_{1i}, X_{2i}, \cdots, X_{hi}$ are levels of factor variables $F_1, F_2, F_3, \cdots, F_h$. Equations (1.6a-c) are commonplace functions in factorial treatment designs and analyses.

When the values in the X matrix are transformed, a type 4 equation results. Consider the example when $X = \sqrt{t}$. The type 1 regression equation $Y_i = A_0 + A_1 X + A_2 X^2 + A_3 X^3$ becomes $Y_i = A_0 + a_1 \sqrt{t} + A_2 t^{3/2}$, a type 4 equation.

This investigation concentrated on the type 1 polynomials. In particular, Legendre, Hermite, Chebyshev (1st kind) and Laguerre were studied. It is shown that the fitted equation for all of the type 1 equations gives the same fitted curve, and that all contrast matrices derived from type 1 equations are identical. It is noted that each orthogonal polynomial is derived from a different differential equation.

2. Type 1 Polynomials

The condition of orthogonality differs for Legendre, Chebyshev, Hermite and Laguerre polynomials. However, in general, the definition of orthogonality requires that when the function is integrated and evaluated over a specific interval, the result is zero. This may be summarized as $\int_a^b P_h(X_i)P_j(X_i)dx = 0$. Some polynomials require that a weight function $\varphi(X)$ be introduced for the definition of orthogonality to hold. In this case, the condition is summarized as $\int_a^b P_h(X_i)P_j(X_i)\varphi(X)dx = 0$. The properties of the orthogonal polynomials considered are given in Appendix A, while the explicit expressions for each are given below.

Legendre

$$P_h(X) = \sum_{j=0}^{h/2} (-1)^j \frac{(2h-2j)!}{2^h j!(h-j)!(h-2j)!} X^{h-2j}$$

Hermite

$$P_h(X) = h! \sum_{j=0}^{h/2} (-1)^j \frac{2^{h-2j}}{j!(h-2j)!} X^{h-2j}$$

Chebyshev (First Kind)

$$P_h(X) = \frac{h}{2} \sum_{j=0}^{h/2} (-1)^j \frac{(h-j-1)!}{j!(h-2j)!} (2X)^{h-2j}$$

Generalized Laguerre

$$P_h^{(\alpha)}(X) = \sum_{j=0}^{h} (-1)^j \binom{h-\alpha}{h-j} \frac{1}{j!} X^j \quad .$$

The general form of the above polynomials is written as

$$P_h(X_i) = \sum_{\ell=0}^{h} a_{h,h-\ell} X_i^{h-\ell} \qquad (2.1)$$

where $P_h(X_i)$ is a polynomial of degree h. The i notation is introduced for extension to statistical data analysis methods.

3. Fitting a Regression Equation

Standard polynomial regression in statistical literature assumes a model of the form of equation (1.2). It is possible that in certain situations a more appropriate model, which gives a better fit, could be obtained by the use of orthogonal polynomials. Consideration of such a model in the form of (1.4), where $P_j(X_i)$ can be any of the orthogonal polynomials considered, or another type 1 polynomial, leads to the following theorem:

Theorem 3.1. Given a set of Y_i observations, the curves fitted by

$$Y_i = f(X_i) = b_0 X_i^0 + b_1 X_i^1 + b_2 X_i^2 + \cdots + b_h X_i^h$$

and

$$Y_i = P(X_i) = A_0 p_0(X_i) + A_1 p_1(X_i) + A_2 p_2(X_i) + \cdots + A_h p_h(X_i)$$

are identities, with the b_j regression coefficients being linear combinations of the A_j regression coefficients such that $b_k = \Sigma_{k=j}^{h} A_j a_{jk}$ for $k = 0,1,2,\cdots,h$.

Proof:

Let S be the set of natural numbers for which Theorem 3.1 is true. Consider h = 1. The standard polynomial regression equation is

$$Y_i = f(X_i) = b_0 X_i^0 + b_1 X_i^1$$

and for k = 0,1

$$b_k = \sum_{k=j}^{1} A_j a_{jk}$$

would imply that

$$b_0 = A_0 a_{00} + A_1 a_{10} \quad \text{and} \quad b_1 = A_1 a_{11} \ .$$

Substituting these in the standard regression equation yields

$$f(X_i) = (A_0 a_{00} + A_1 a_{10})X_i^0 + A_1 a_{11} X_i^1$$

$$= A_0 a_{00} + A_1 (a_{11} X_i^1 + a_{10})$$

$$= A_0 p_0(X_i) + A_1 p_1(X_i) = P_1(X_i) \ .$$

Therefore $1 \in S$.

Assume Theorem 3.1 holds for $h = m$. Then for $k = 1,2,3,\cdots,m$, $b_k = \Sigma_{k=j}^{m} A_j a_{jk}$ gives the following result:

$$f_m(X_i) = b_0 X_i^0 + b_1 X_i^1 + b_2 X_i^2 + \cdots + b_m X_i^m$$

$$= \Big(\sum_{j=0}^{m} A_j a_{j0}\Big)X_i^0 + \Big(\sum_{j=1}^{m} A_j a_{j1}\Big)X_i^1 + \Big(\sum_{j=2}^{m} A_j a_{j2}\Big)X_i^2 + \cdots + \Big(\sum_{j=m}^{m} A_j a_{jm}\Big)X_i^m$$

$$= A_0(a_{00}X_i^0) + A_1\Big(\sum_{\ell=0}^{1} a_{1,1-\ell} X_i^{1-\ell}\Big) + A_2\Big(\sum_{\ell=0}^{2} a_{2,2-\ell} X_i^{2-\ell}\Big) + \cdots$$

$$+ A_m\Big(\sum_{\ell=0}^{m} a_{m,m-\ell} X_i^{m-\ell}\Big) = P_m(X_i) \ .$$

To consider $h = m + 1$, the appropriate terms are added.

$$f_{m+1}(X_i) = b_0 X_i^0 + b_1 X_i^1 + b_2 X_i^2 + \cdots + b_m X_i^m + b_{m+1} X_i^{m+1}$$

$$= \Big(\sum_{j=0}^{m} A_j a_{j0} + A_{m+1} a_{m+1,0}\Big)X_i^0 + \Big(\sum_{j=1}^{m} A_j a_{j1} + A_{m+1} a_{m+1,1}\Big)X_i^1$$

$$+ \left(\sum_{j=2}^{m} A_j a_{j2} + A_{m+1} a_{m+1,2}\right) X_i^2 + \cdots$$

$$+ \left(\sum_{j=m}^{m} A_m a_{mm} X_i^m + A_{m+1} a_{m+1,m} X_i^m\right) + A_{m+1} a_{m+1} X_i^{m+1}$$

$$= \left(\sum_{j=0}^{m+1} A_j a_{j0}\right) X_i^0 + \left(\sum_{j=1}^{m+1} A_j a_{j1}\right) X_i^1 + \left(\sum_{j=2}^{m+1} A_j a_{j2}\right) X_i^2 + \cdots$$

$$+ \left(\sum_{j=m}^{m+1} A_j a_{jm}\right) X^m + A_{m+1} a_{m+1} X^{m+1}$$

$$= A_0 (a_{00} X_i^0) + A_1 \left(\sum_{\ell=0}^{1} a_{1,1-\ell} X_i^{1-\ell}\right) + A_2 \left(\sum_{\ell=0}^{2} a_{2,2-\ell} X_i^{2-\ell}\right) + \cdots$$

$$+ A_{m+1} \left(\sum_{\ell=0}^{m+1} a_{m+1,m+1-\ell} X_i^{m+1}\right)$$

$$= P_{m+1}(X_i) \quad .$$

Therefore if $m \in S$, then $m+1 \in S$. Therefore $S = N = \{1,2,3,\cdots\}$ and Theorem 3.1 holds for the set of natural numbers.

Although only type 1 polynomials were considered here, further attention should be given to the other types. Theorem 3.1 does not hold for these polynomials, and they may provide useful alternative models for data analysts.

4. Orthogonal Contrast Matrices

Single degree of freedom contrasts are important in data analysis, because they are useful in pin-pointing the sources of variation among treatment means in an experiment. A matrix of (v-1) single degree of freedom contrasts for v treatment means is shown below.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2v} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{v1} & c_{v2} & c_{v3} & \cdots & c_{vv} \end{bmatrix} = \underset{\sim}{C} = [c_{hi}] \quad . \tag{4.1}$$

The first row of ones represents the common effect or the overall mean for the v treatments. When $\Sigma_{i=1}^{v} c_{hi}c'_{hi} = 0$ for $h \neq h' = 1,2,3,\cdots,v$ and $\Sigma_{i=1}^{v} c_{hi}^2 = 1$ for all h, then $C_{vXv}C'_{vXv} = I_{vXv}$ and C_{vXv} is called an orthogonal normalized contrast matrix.

There are several common classes of orthogonal contrast matrices. One class consists of the Helmert matrices whose general form is given below.

$$\begin{bmatrix} \frac{1}{\sqrt{v}} & \frac{1}{\sqrt{v}} & \frac{1}{\sqrt{v}} & \cdots & \frac{1}{\sqrt{v}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \cdots & 0 \\ \frac{1}{\sqrt{v(v-1)}} & \frac{1}{\sqrt{v(v-1)}} & \frac{1}{\sqrt{v(v-1)}} & \cdots & \frac{1}{\sqrt{v(v-1)}} \end{bmatrix} \tag{4.2}$$

Another class of orthogonal contrast matrices, the Hadamard matrices, are composed entirely of plus and minus ones. They exist only for v = 4t and are used extensively to formulate single degree of freedom contrasts in 2^h factorial experiments. Several examples follow.

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{4.3}$$

The class considered in most detail here is that which is obtained by applying the Gram-Schmidt orthogonalization procedure mentioned in section 1 to the X matrix (1.3), resulting in the orthogonal polynomial matrix. The general form of this class of matrice is given as

$$\underline{r} = \begin{bmatrix} 1 & r_{11} & r_{21} & \cdots & r_{h1} \\ 1 & r_{12} & r_{22} & \cdots & r_{h2} \\ 1 & r_{13} & r_{23} & \cdots & r_{h3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{1n} & r_{2n} & \cdots & r_{hn} \end{bmatrix} \tag{4.4}$$

The columns are obtained by applying the formulas given below to the appropriate columns of the X matrix (1.3).

$$r_{1i} = X_i - \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$r_{2i} = X_i^2 - \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{r_{1i} \sum_{i=1}^{n} r_{1i} X_i^2}{\sum_{i=1}^{n} r_{1i}^2}$$

$$r_{3i} = X_i^3 - \frac{1}{n}\sum_{i=1}^{n} X_i^3 - \frac{r_{1i} \sum_{i=1}^{n} r_{1i} X_i^3}{\sum_{i=1}^{n} r_{1i}^2} - \frac{r_{2i} \sum_{i=1}^{n} r_{2i} X_i^3}{\sum_{i=1}^{n} r_{2i}^2} .$$

The pattern is apparent and is generalized as

$$r_{ji} = X_i^j - \frac{1}{n}\sum_{i=1}^{n} X_i^j - \sum_{\ell=1}^{j-1} \left[\frac{r_{\ell i} \sum_{i=1}^{n} r_{\ell i} X_i^j}{\sum_{i=1}^{n} r_{\ell i}^2} \right] . \qquad (4.5)$$

This formula applies for both equally and unequally spaced X_i. The results of the Gram-Schmidt process for equally spaced X_i are shown in Fisher and Yates (1963), Table XXIII (n or v = 3 to 52).

The use of the Legendre, Hermite, Chebyshev and Laguerre equations was investigated by applying the Gram-Schmidt orthogonalization procedure to a different matrix

$$\underline{X}^* = \begin{bmatrix} 1 & p_1(X_1) & p_2(X_1) & \cdots & p_h(X_1) \\ 1 & p_1(X_2) & p_2(X_2) & \cdots & p_h(X_2) \\ 1 & p_1(X_3) & p_2(X_3) & \cdots & p_h(X_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_1(X_h) & p_2(X_h) & \cdots & p_h(X_h) \end{bmatrix} . \qquad (4.6)$$

The columns of X^* were substituted in equation (4.5) for the X_i terms and the result is summarized as shown.

$$R_{ji} = P_j(X_i) - \frac{1}{n}\sum_{i=1}^{n} P_j(X_i) - \sum_{\ell=1}^{j-1}\left[\frac{r_{\ell i}\ \sum_{i=1}^{n} r_{\ell i} P_j(X_i)}{\sum_{i=1}^{n} r_{\ell i}^2}\right] \quad . \quad (4.7)$$

Applying this formula to each column of X* gives the following matrix:

$$\underline{R} = \begin{bmatrix} 1 & R_{11} & R_{21} & \cdots & R_{h1} \\ 1 & R_{12} & R_{22} & \cdots & R_{h2} \\ 1 & R_{13} & R_{23} & \cdots & R_{h3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & R_{1n} & R_{2n} & \cdots & R_{hn} \end{bmatrix} \quad .$$

A study of this resulted in Theorem 4.1.

Theorem 4.1. Given X and X*, the Gram-Schmidt orthogonalization process generates the respective orthogonal contrast matrices $\underline{r}$ and $\underline{R}$, such that $R_{ji} = a_{jj} r_{ji}$.

Proof: Let S be the set of natural numbers k for which $R_{ji} = a_{jj} r_{ji}$.

Consider k = 1.

$$R_{1i} = P_1(X_i) - \frac{1}{n}\sum_{i=1}^{n} P_1(X_i)$$

$$= a_{11}X_i + a_{10} - \frac{1}{n}\sum a_{11}X_i - \frac{na_{10}}{n}$$

$$= a_{11}\left(X_i - \frac{1}{n}\sum_{i=1}^{n} X_i\right) = a_{11}r_i \quad .$$

Therefore $1 \in S$.

Consider $k = 2$.

$$R_{2i} = P_2(X_i) - \frac{1}{n}\sum_{i=1}^{n} P_2(X_i) - \frac{r_{1i}\sum_{i=1}^{n} r_{1i}P_2(X_i)}{\sum_{i=1}^{n} r_i^2}$$

$$= (a_{22}X_i^2 + a_{21}X_i + a_{20}) - \frac{1}{n}\left(\sum_{i=1}^{n} a_{22}X_i^2 + a_{21}X_i + a_{20}\right)$$

$$- \frac{r_{1i}\sum_{i=1}^{n} r_{1i}(a_{22}X_i^2 + a_{21}X_i + a_{20})}{n}$$

$$= a_{22}\left[X_i^2 - \frac{\sum_{i=1}^{n} X_i^2}{n} - \frac{r_{1i}\sum_{i=1}^{n} r_{1i}X_i^2}{\sum_{i=1}^{n} r_i^2}\right]$$

$$+ a_{21}\left[X_i - \frac{\sum_{i=1}^{n} X_i}{n} - \frac{r_{1i}\sum_{i=1}^{n} r_{1i}X_i}{\sum_{i=1}^{n} r_{1i}^2}\right] .$$

Note that

$$\frac{\sum_{i=1}^{n} r_{hi}X_i^m}{\sum_{i=1}^{n} r_i^2} = \begin{cases} 1, & m = h \\ 0, & m < h \end{cases} .$$

Therefore, $R_{2i} = a_{22}(r_{2i}) + a_{21}(r_{1i} - r_{1i}) = a_{22}r_{2i}$. Therefore, $2 \in S$.

Consider $k = m$ and assume $R_{ji} = a_{jj}v_{ji}$ holds, Then,

$$R_{mi} = P_m(X_i) - \frac{1}{n}\sum_{i=1}^{n} P_m(X_i) - \sum_{\ell=1}^{m-1}\left[\frac{r_{\ell i}\sum_{i=1}^{n} r_{\ell i}P_m(X_i)}{\sum_{i=1}^{n} r_{\ell i}^2}\right]$$

$$= a_{mm}\left[X_i^m - \frac{\sum_{i=1}^{n} X_i^m}{n} - \sum_{\ell=1}^{m-1}\left(\frac{r_{\ell i}\Sigma r_{\ell i}X_i^{m-1}}{\sum_{i=1}^{n} r_{\ell i}^2} \right)\right]$$

$$+ a_{m,m-1}\left[X_i^{m-1} - \frac{\sum_{i=1}^{n} X_i^{m-1}}{n} - \sum_{\ell=1}^{m-1}\left(\frac{r_{\ell i} \sum_{i=1}^{n} r_{\ell i}X_i^{m-1}}{\sum_{i=1}^{n} r_i^2} \right)\right]$$

$$+ a_{m,m-2}\left[X_i^{m-2} - \frac{\sum_{i=1}^{n} X_i^{m-2}}{n} - \sum_{\ell=1}^{m-1}\left(\frac{r_{\ell i} \sum_{i=1}^{n} r_{\ell i}X_i^{m-1}}{\sum_{i=1}^{n} r_i^2} \right)\right] + \cdots$$

$$+ a_{m,0}\left[X_i^{0} - \frac{\sum_{i=1}^{n} X_i^{0}}{n} - \sum_{\ell=1}^{m-1}\left(\frac{r_{\ell i} \sum_{i=1}^{n} r_{\ell i}}{\sum_{i=1}^{n} r_{\ell i}^2} \right)\right]$$

$$= a_{mm}v_{mi} + a_{m,m-1}(r_{m-1,i} - r_{m-1,i}) + a_{m,m-2}(r_{m-2,i} - r_{m-2,i}) + \cdots$$

$$+ a_{m,1}(r_{1i} - r_{1i})$$

$$= a_{mm}v_{mi} \quad .$$

Now add the appropriate term for $k = m + 1$. The above equations become the following:

$$R_{m+1,i} = a_{m+1,m+1}\left[X^{m+1} - \frac{\sum_{i=1}^{n} X^{m+1}}{n} - \sum_{\ell=1}^{m}\left(\frac{r_{\ell i} \sum_{i=1}^{n} r_{\ell i}X^{m+1}}{\sum_{i=1}^{n} r_{\ell i}^2} \right)\right]$$

$$+ a_{m+1,m}\left[r_{mi} - \frac{r_{mi} \sum_{i=1}^{n} r_{mi} X_i^m}{\sum_{i=1}^{n} r_{mi}^2} \right]$$

$$+ a_{m+1,m-1}\left[r_{m-1,i} - r_{m-1,i} - \frac{r_{mi} \sum_{i=1}^{n} r_{mi} X^{m-1}}{\sum_{i=1}^{n} r_{mi}^2} \right]$$

$$+ a_{m+1,m-2}\left[r_{m-2,i} - r_{m-2,i} - \frac{r_{mi} \sum_{i=1}^{n} r_{mi} X^{m-2}}{\sum_{i=1}^{n} r_{mi}^2} \right] + \cdots$$

$$+ a_{m,1}\left[r_{1i} - r_{1i} - \frac{r_{mi} \sum_{i=1}^{n} r_{mi} X}{\sum_{i=1}^{n} r_{mi}^2} \right]$$

$$= a_{m+1,m+1} r_{m+1,i} \quad .$$

Therefore, if $m \in S$, $m+1 \in S$. It is concluded that Theorem 4.1 holds for $S = N = \{1,2,3,\cdots\}$, the set of natural numbers.

The result of this theorem is important because it explains why all polynomials of type 1 produce identical contrast matrices after the columns have been divided by a_{jj}.

No new areas of applications were found for the orthogonal polynomials considered, but it is clear that further investigation is needed in the examination of such functions and their possible use in statistics. Here we only mentioned type 2, 3 and 4 polynomials. A study of these may produce results useful in the statistical analysis of data.

APPENDIX A

Orthogonal Polynomials

A. Legendre

Interval: $[-1,1]$ Weight: $1 = \varphi(X)$

Orthogonality Relation:

$$\int_{-1}^{1} P_h(X)P_j(X)dx = \begin{cases} 0 & , \quad h \neq j \\ \dfrac{2}{2h+1} & , \quad h = j \end{cases}$$

Differential Equation:

$$(1-X^2)P_h'' - 2XP_h' + h(h+1)P_h = 0$$

Recurrence Relation:

$$(h+1)P_{h+1}(X) = (2h+1)XP_h(X) - hP_{h-1}(X)$$

Rodrigues Formula:

$$P_h(X) = \frac{(-1)^h}{2^h h!} \frac{d^h}{dX^h} \{(1-X^2)^h\}$$

Generating Function:

$$(1 - 2XZ + Z^2)^{-\frac{1}{2}} = \sum_{h=0}^{\infty} P_h(X)Z^h \; ; \quad -1 < X < 1, \quad |Z| < 1$$

B. Hermite

Interval: $(-\infty,\infty)$ Weight: $e^{-X^2} = \varphi(X)$

Orthogonality Relation:

$$\int_{-\infty}^{\infty} e^{-X^2} P_j(X)P_h(X)dx = \begin{cases} 0 & , \quad h \neq j \\ \pi^{\frac{1}{2}}2^h h! & , \quad h = j \end{cases}$$

Differential Equation:

$$P_h'' - 2XP_h' + 2hP_h = 0$$

Recurrence Relation:

$$P_{h+1}(X) - 2XP_h(X) + 2hP_{h-1}(X) = 0$$

Rodrigues Formula:

$$P_h(X) = (-1)^h e^{X^2} \frac{d^h}{dX^h} (e^{-X^2})$$

Generating Function:

$$\exp(2ZX - Z^2) = \sum_{h=0}^{\infty} \frac{1}{h!} P_h(X)Z^h \quad ; \quad |Z| < 1$$

C. <u>Chebyshev</u> (<u>Tscheysheff</u>) <u>The</u> <u>First</u> <u>Kind</u>

Interval: $[-1,1]$ Weight: $(1-X^2)^{-\frac{1}{2}} = \varphi(X)$

Orthogonality Relation:

$$\int_{-1}^{1} P_j(X)P_h(X)(1-X^2)^{-\frac{1}{2}}dx = \begin{cases} 0 & , \quad h \neq j \\ \pi/2 & , \quad h = j \neq 0 \\ \pi & , \quad h = j = 0 \end{cases}$$

Differential Equation:

$$(1 - X^2)P_h'' - XP_h' + h^2P_h = 0$$

Recurrence Relation:

$$P_{h+1}(X) = 2XP_h(X) - P_{h-1}$$

Rodrigues Formula:

$$\frac{(-1)^h (1 - X^2)^{\frac{1}{2}}\sqrt{\pi}}{2^{h+1}\Gamma(h + \frac{1}{2})} \frac{d^h}{dX^h} \left[(1 - X^2)^{h-\frac{1}{2}}\right] = P_h(X)$$

Generating Function:

$$\frac{1 - XZ}{1 - 2XZ + Z^2} = \sum_{h=0}^{\infty} P_h(X)Z^h \quad ; \quad -1 < X < 1 \, , \quad |Z| < 1$$

D. Generalized Laguerre

Interval: $[0,\infty]$ Weight: $X^{\alpha}e^{-X} = \varphi(X)$; $\alpha > -1$

Orthogonality Relation:

$$\int_0^{\infty} e^{-X}X^{\alpha}P_j^{(\alpha)}(X)P_h^{(\alpha)}(X)dx = \begin{cases} 0 & , \quad h \neq j \\ \Gamma(1+\alpha)\binom{h+\alpha}{h} & , \quad h = j \end{cases}$$

Differential Equation:

$$XP_h^{(\alpha)''} + (\alpha + 1 - X)P_H^{(\alpha)'} + hP_h^{(\alpha)} = 0$$

Recurrence Relation:

$$(h + 1)P_{h+1}^{(\alpha)}(X) = [(2h + \alpha + 1) - X]P_h^{(\alpha)}(X) - (h + \alpha)P_{h-1}^{(\alpha)}(X)$$

Rodrigues Formula:

$$P_h^{(\alpha)}(X) = \frac{1}{h!X^{\alpha}e^{-X}} \frac{d^h}{dX^h}\left\{X^{h+\alpha}e^{-X}\right\}$$

Generating Function:

$$(1 - Z)^{-\alpha-1} \exp\left(\frac{XZ}{Z - 1}\right) = \sum_{h=0}^{\infty} P_h^{(\alpha)}(X)Z^h$$

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